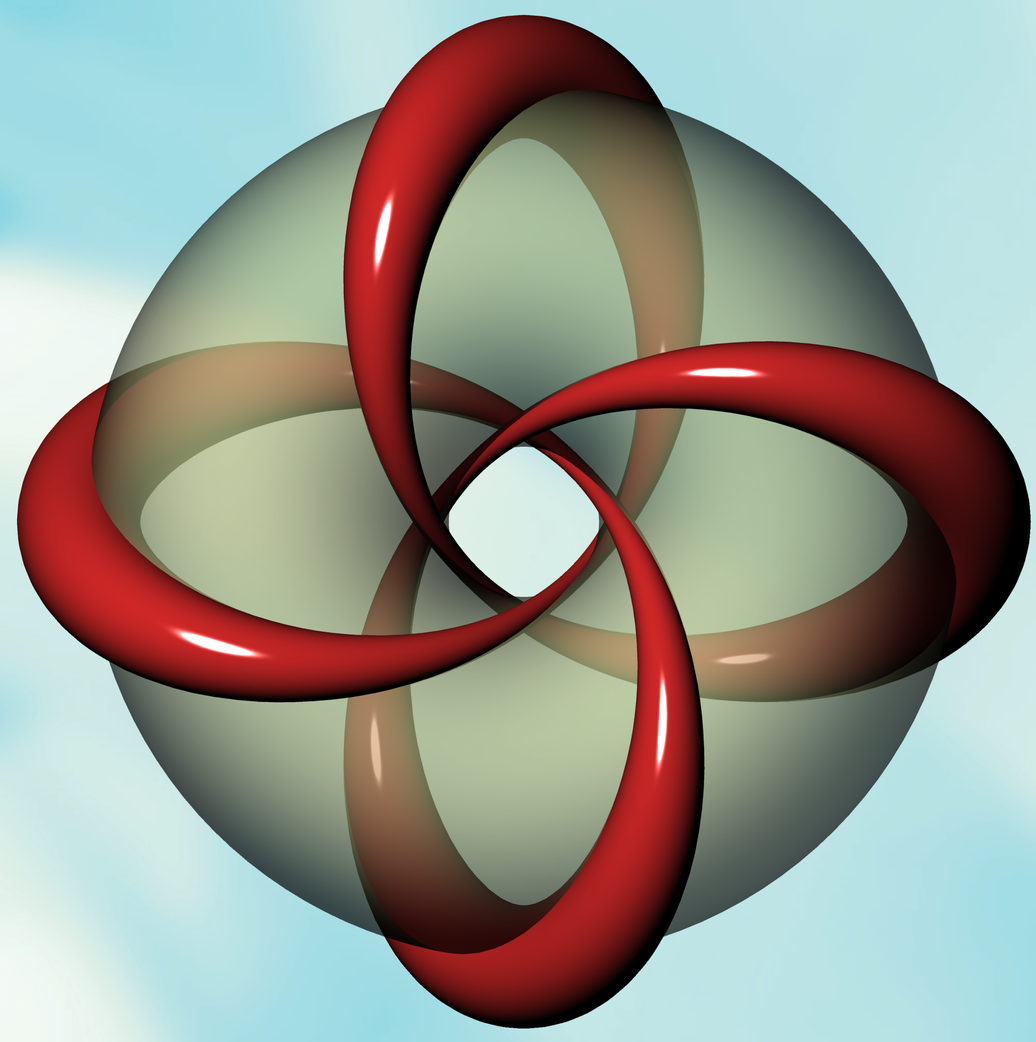
Seifert surfaces

Torus knots

Ordinary knots, like shoelaces, can be tied and untied. To make them more topologically interesting we join the ends together. This gives the mathematician's *knot*: a closed loop in three-space.

The best known of these is the *trefoil* knot, obtained by joining the ends of an overhand knot. The trefoil is the first of the *torus knots*: knots that can be arranged to lie on a *torus* (the grey donut to the right). Surprisingly there is one such knot for every rational number p/q. Here p records the number of times the knot goes through the hole of the torus while q records the number of times it goes around the torus. The thick red knot shown to the right corresponds to $4/3$. (Challenge: Draw the trefoil, with fraction $3/2$, on the torus.)



Knots, surfaces, three-manifolds

A knot K is one-dimensional; in and of itself it is just a loop. A *Seifert surface* for the knot K gives a two-dimensional way to understand how K lies inside of three-space.

Figures 9–11 from *Über das Geschlecht von Knoten* (1934), by Herbert Seifert, give three views of the $(7, -3, 5)$ pretzel knot. Each figure also shows a Seifert surface S: the back of S is indicated by the horizontal shading. In the first figure S is shown as a disk with two writhing bands attached to it. In the second figure, the outer writhing has been converted to twisting. In the third figure the central region has been rotated, deforming S into a pair of disks (above and below) with three twisted bands connecting them. Note how all of the manipulations are *continuous*: the knot K and surface S never crash through themselves or through each other.

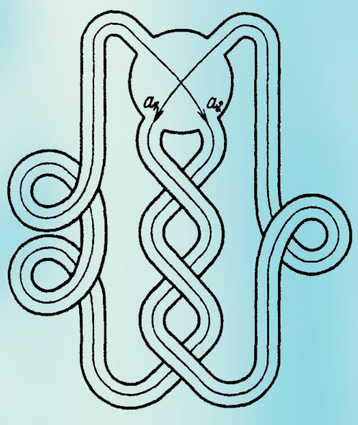


Fig. 9.

Fig. 10.

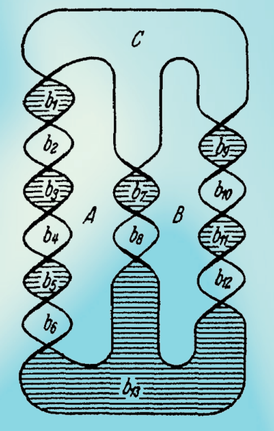
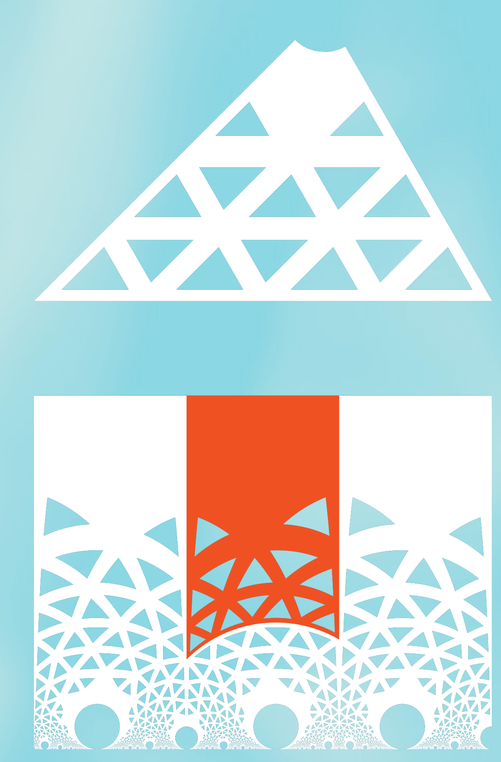


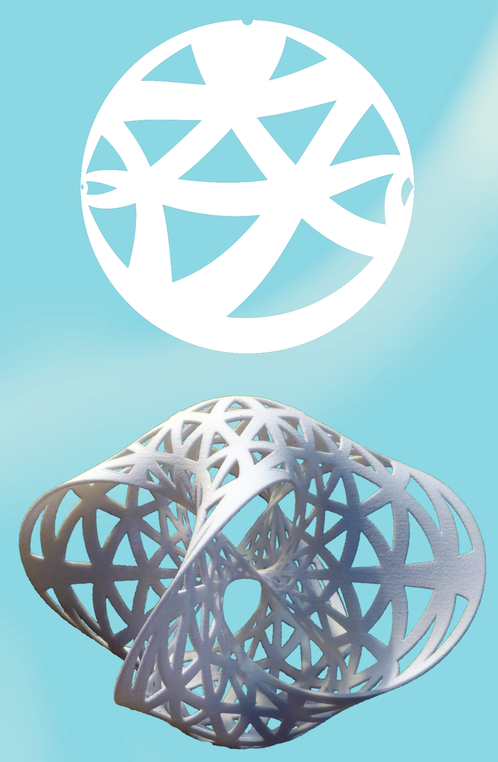
Fig 11.

Illustrating Milnor fibers

Torus knots, by their very definition, have a best position. To find it, we place the torus optimally in the three-sphere S^3 to obtain the Clifford torus C, shown in grey above. The torus C intrinsically has the geometry of a flat square with its sides identified. The p/q torus knot K lies in C as a straight line lies in the plane. In a wonderful agreement between geometry and algebra, K is also cut out of S^3 by the equation $z^p + w^q = 0$ where z and w are complex numbers.

Milnor gives canonical Seifert surfaces for K: namely the surfaces S_θ cut out of S^3 by the equation $\arg(z^p + w^q) = \theta$. We parametrize S_θ in a sequence of steps – as shown to the right we go from a Euclidean triangle, to the Poincare disk, to a hyperbolic triangle (in red), to a surface in S^3. This is stereographically projected into three-space, giving our final image.





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